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NOTE

Small Entire Functions with Infinite Growth Index

A. Bonilla[1]

Departamento de Análisis Matemático, Universidad de La Laguna,
38271 La Laguna (Tenerife), Spain
E-mail: abonilla@ull.es

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In this paper, we prove that given $\mu > 0$ there exists a dense linear manifold M of entire functions, such that,

$$\lim_{\substack{z\to\infty \\ z\in l}} \exp(|z|^{\mu}) f(z) = 0,$$

for every $f \in M$ and l straight line and with infinite growth index for all non-null functions of M. Moreover, every non-null function of M has exactly $2([2\mu]+1)$ Julia directions. And if l is a straight line that does not contain a Julia line, then for every $f \in M$

$$\lim_{\substack{z\to\infty \\ z\in l}} \exp(|z|^{\mu}) f^{(j)}(z) = 0,$$

and for $j \geq 1$, $f^{(j)}$ is bounded and integrable with respect to the length measure on l and $\int_l f^{(j)} = 0$.

1. INTRODUCTION

Bernal [7] proved that, given $\mu \in (0, \frac{1}{2})$, there exists a linear manifold M of entire functions, which is dense in the space of all entire functions and, in addition,

$$\lim_{\substack{z\to\infty \\ z\in S}} \exp(|z|^{\mu}) f^{(j)}(z) = 0$$

for every $f \in M$ and $j \in \mathbb{N}$, where S denotes any plane strip.

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As a consequence, for $j \geq 1$, $f^{(j)}$ is integrable on every straight line with respect to length measure s, and

$$\int_l f^{(j)}\, ds = 0.$$

On the other hand, the growth index is infinite for all non-null functions of M.

The main tool of the proof of this result is the Arakelyan's theorem about approximation with speed by entire functions. But, this theorem says nothing of the case where $\mu \geq \frac{1}{2}$. However, if $\mu \geq \frac{1}{2}$, Armitage and Goldstein using a result of harmonic approximation proved that there exists a non-null entire function f such that

$$\lim_{\substack{z\to\infty \\ z\in l}} \exp(|z|^{\mu}) f(z) = 0. \tag{1}$$

Moreover, if (1) is true for all positive numbers μ, then $f \equiv 0$ (see [4]).

The problem that we are concerned with in this paper is to prove a result similar to [7] for $\mu \geq \frac{1}{2}$. That is, given $\mu > 0$ there exists a dense linear manifold M of entire functions, such that

$$\lim_{\substack{z\to\infty \\ z\in l}} \exp(|z|^{\mu}) f(z) = 0,$$

for every $f \in M$ and l straight line and with infinite growth index for all non-null functions of M.

Moreover, every non-null function of M has exactly $2([2\mu] + 1)$ Julia directions. And if l is a straight line that does not contain a Julia line, then for every $f \in M$, we have that

$$\lim_{\substack{z\to\infty \\ z\in l}} \exp(|z|^{\mu}) f^{(j)}(z) = 0.$$

And as a consequence, $f^{(j)}$ is bounded and integrable for $j \geq 1$ with respect to the length measure on l and $\int_l f^{(j)} = 0$.

2. RESULTS

Denote $H(\mathbb{C})$ as the space of all entire functions, endowed with the compact–open topology, and denote $\mathbb{C}^*$ as the one-point compactification of $\mathbb{C}$. If F is a closed subset of $\mathbb{C}$, then $A(F)$ is the set of all continuous functions on F and holomorphic on interior of F.

Given f as an entire function, if $r > 0$, we define $\exp_1 r := \exp r$, $\exp_{m+1} r := \exp(\exp_m r)$ $(m \in \mathbb{N})$, and $M_f(r) := \max\{|f(z)| : |z| = r\}$.

For $r > 0$ large enough, we denote $\log_1 r := \log r$, $\log_{m+1} := \log(\log_m r)$ ($m \in \mathbb{N}$). The m-order growth $\rho_m = \rho_m(f)$ of f is defined to be

$$\rho_m := \limsup_{r\to\infty} \frac{\log_{m+1} M_f(r)}{\log r}.$$

Note that ρ_1 is the ordinary growth order of f. The growth index of f is $i(f) := \min\{m \in \mathbb{N} : \rho_m(f) < \infty\}$, and we set $i(f) = \infty$ when $\rho_m(f) = \infty$ for all m.

We call ϕ a Julia direction of the entire function f, if in every sector $\{z : |\arg z - \phi| < \delta\}$ with $\delta > 0$, the function f assumes each complex value infinitely often, with at most one exception. And the ray $\{re^{i\theta} : r \in \mathbb{R}^+\}$ is a Julia line. Denote $J(f) := \{e^{i\phi} : \phi \text{ is a Julia direction of } f\}$.

In the proof of the main result, we need the following lemma:

LEMMA 1 [1, Lemma 6]. *Let F be a closed subset of the complex plane such that $\mathbb{C}^*\backslash F$ is connected and locally connected at $\{*\}$, and w a continuous, bounded, zero-free function on F, holomorphic on F^o. Then, given $g \in A(F)$, there exists an entire function h such that*

$$|g(z) - h(z)| < |w(z)| \quad z \in F.$$

We are now ready to state our results.

THEOREM 1. *Given $\mu > 0$, there exists a dense linear manifold M of entire functions, such that:*

(a) $\lim_{\substack{z\to\infty \\ z\in l}} \exp(|z|^\mu) f(z) = 0$ *for every $f \in M$ and l straight line.*

(b) $i(f) = \infty$, *for all non-null functions of M.*

(c) *Every non-null function of M has exactly* $2([2\mu] + 1)$ *Julia directions.*

(d) *If l is a straight line that does not contain a Julia line, then for every $f \in M$*

(i) $\lim_{\substack{z\to\infty \\ z\in l}} \exp(|z|^\mu) f^{(j)}(z) = 0$

(ii) *$f^{(j)}$ is bounded and integrable with respect to the length measure on l and $\int_l f^{(j)} = 0$.*

Proof. Consider a denumerable family of polynomials $\{p_n\}$ dense in $H(\mathbb{C})$ and let $n = [2\mu] + 1$. Denote $T(\alpha) = \{re^{i\theta} : r > 0, |\theta| < \alpha\}$ and define

$$G_j := \left\{ ze^{ij\frac{\pi}{n}} : z \in T\left(\frac{\pi}{2n}\right) \right\} \quad (j = 0, 1, \ldots, 2n-1),$$

$$G := G_0 \cup \cdots \cup G_{2n-1}$$

$$K := \{re^{i\theta} : r^n |\cos n\theta| \geq 1\}.$$

We can observe that K is a closed set with $2n$ connected components K_j such that $K_j \subset G_j$ ($j = 0, 1, \ldots, 2n-1$). Also define $L := \partial G \backslash D(0, 1)$, so that L consists of $2n$ disjoint semi-infinite lines. Then there exists h as an entire function, bounded in $K \cup L$, zero-free, and

$$\exp(r^{\mu}) h(a + re^{i\theta}) \to 0 \quad (r \to \infty),$$

for all complex numbers a and all real numbers θ ([4, Example 7]).

Let $B_n := \{z : |z| < n\}$ and let $E_n := (K \cup L) \backslash B_{n+1}$. Moreover we choose a sequence $\{z_k\}_{k \in \mathrm{N}} \subset (K \cup L)^c$ so that it has a point in each component of $(K \cup L)^c$ of the same modulus and $r_k = |z_k|$ is increasing.

Consider

$$F_n := \overline{B_n} \cup E_n \cup \{z_k\}_{k \in \mathrm{N}}.$$

Note that F_n is a closed subset of $\mathbb{C}$, and $\mathbb{C}^* \backslash F_n$ is connected and locally connected at $\{*\}$, and if we define

$$w_n(z) := \begin{cases} \dfrac{1}{n} & z \in \overline{B_n} \\ h(z) & z \in E_n \\ \dfrac{1}{n} & z = z_k \text{ and } |z_k| > n, \end{cases}$$

then w_n satisfies the hypothesis of Lemma 1 on F_n.

Now we define the function $g_1 \colon F_1 \to \mathbb{C}$ by

$$g_1(z) := \begin{cases} p_1(z) & z \in \overline{B_1} \\ 0 & z \in E_1 \\ 1 + \exp_{k+1} r_k & z = z_k \text{ and } |z_k| > 1. \end{cases}$$

Then $g_1 \in A(F_1)$, and by Lemma 1, there exists an entire function f_1 such that

$$|(g_1 - f_1)(z)| < |w_1(z)|, \ \ z \in F_1. \tag{2}$$

Assume that $n \in \{2, 3, \ldots\}$ and that we have constructed $2n-2$ functions $g_1, f_1, \ldots g_{n-1}, f_{n-1}$ such that $g_i \in A(F_i)$, $f_i \in H(\mathbb{C})$, and $|(g_i - f_i)(z)| < |w_i(z)|$, $z \in F_i$, $\forall\, i \in 1, 2, \ldots n-1$. And we define the function $g_n : F_n \to \mathbb{C}$ by

$$g_n(z) := \begin{cases} p_n(z) & z \in \overline{B_n} \\ 0 & z \in E_n \\ 1 + \exp_{k+1} r_k + k \displaystyle\sum_{i=1}^{n-1} M_{f_i}(r_k) & z = z_k \text{ and } |z_k| > n. \end{cases}$$

Then $g_n \in A(F_n)$, and by Lemma 1, there exists an entire function f_n such that

$$|(g_n - f_n)(z)| < |w_n(z)|, \ z \in F_n. \tag{3}$$

Hence

$$|(g_n - f_n)(z)| < \frac{1}{n}, \ z \in B_n.$$

Thus the sequence $\{f_n\}_{n=1}^{\infty}$ is dense in $H(\mathbb{C})$. Now, we define M as the linear span of $\{f_n\}$. Evidently, M is a linear dense manifold of $H(\mathbb{C})$. To verify (a), it is sufficient to check it with f_n. If $a \in \mathbb{C}$ and $\theta \in \mathbb{R}$, then it is easy to see that $a + re^{i\theta} \in E_n$ for r large enough, and when $r \to +\infty$

$$|f_n(a + re^{i\theta})| \le |h(a + re^{i\theta})| = o(e^{-r^{\mu}}).$$

We can deduce (b) in analogous form to [7].

To prove (c), if $f \in M \backslash \{0\}$, it is clear that $J(f) \subset \{e^{ij\frac{\pi}{n}} : j = 0, 1, \ldots, 2([2\mu] + 1) - 1\}$. Moreover if $e^{ij\frac{\pi}{n}} \notin J(f)$, then there exists a sector W such that $e^{ij\frac{\pi}{n}} \in W$ in which f assumes two values at most finitely often. Thus the sequence of functions $f_n(z) := f(nz)$ is a normal family in W, and $\{f_n\}$ converges for all $z \in W$, and therefore it converges uniformly on a compact subset of W. But then each subsector W' of W has the property that f is bounded or $\frac{1}{f}$ is bounded; that is not true in this case.

Finally making use of the Cauchy's inequalities and the fundamental calculus theorem, we obtain (d).

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